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REPRESENTATIONS OF MONOIDS ARISING FROM FINITE GROUPS OF LIE TYPE

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Abstract. A class of finite monoids M constructed from a group G of Lie type is considered. We describe the irreducible complex representations and prove the complete reducibility of the representations of M. The sandwich matrix of M is decomposed into a product of matrices corresponding to maximal parabolic subgroups of G.

1. Introduction

Monoids of Lie type were introduced in [7] as finite analogues of linear algebraic monoids. They were used to solve the long-standing problem on the semisimplicity of the complex algebra of the full matrix monoid $M_n(F_q)$ over a finite field F_q, [6]. Their connections with the representation theory and combinatorics of finite groups of Lie type, via so called sandwich matrices, then became apparent and were extensively studied in [9, 10, 12]. In particular, this approach has recently led to a new explicit description of the Steinberg representation, [11] .

In this paper we consider three $\mathcal{J}$-class monoids $M = M(G, P, P^-, L) = G \sqcup J \sqcup \{0\}$ where G is the group of units and any two idempotents in J are conjugate. Moreover we assume that G is a finite group with a split BN-pair satisfying some commutator relations. P, P^- are parabolic subgroups of G with a common Levi factor L. In the case where P and P^- are opposite, these monoids are called universal three $\mathcal{J}$-class monoids of Lie type and give the local structure of any monoid of Lie type, [7, 8]. We prove that the complex semigroup algebra $\mathbf{C}_0[M]$ is semisimple in the general case. This is done by proving that certain $\mathbf{C}[M]$-modules are irreducible, which turns out to be equivalent to showing that some homomorphisms of $\mathbf{C}[G]$-modules are in fact isomorphisms. These homomorphisms were used to construct the standard bases of Hecke algebras in the cuspidal case, [4]. So the semisimplicity problem for $\mathbf{C}_0[M]$ is formulated in terms of group representation theory. Moreover all irreducible representations of M are described explicitly. In the last section certain decomposition of the sandwich matrix of M is obtained. This reduces the problem of finding the inverse of this matrix to the case where P and P^- are maximal and opposite. It is worth mentioning that this case is crucial for the motivating example $M = M_n(F_q)$ which was considered in [5, 6]. In particular this shows that considering our class of monoids, wider than monoids of Lie type, is natural since it allows an induction. Our techniques are built on

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representation theory of finite groups of Lie type. The monograph of Carter, [1], will be the standard reference.

2. Irreducible representations

We briefly review some basics of semigroup theory, see [2] for details. Let M be a monoid (i.e. has an identity element). $\mathcal{J}$ will denote one of Green's equivalence relations on M : $a\mathcal{J}b$ if $MaM = MbM$. If J is a $\mathcal{J}$-class of M, then define $J^0 = J \cup \{0\}$ with

$$a \cdot b = \begin{cases} ab & \text{if } ab \in J, \\ 0 & \text{otherwise.} \end{cases}$$

If M is finite then J^0 is either a null semigroup or else a completely 0-simple semigroup. Any completely 0-simple semigroup has a Rees representation $\mathcal{M}(H, I_1, I_2, \mathcal{P})$ where H is a maximal subgroup and $\mathcal{P}$ is an $I_2 \times I_1$ sandwich matrix with entries in $H \cup \{0\}$.

Let G be a finite group with subgroups P, P^- and homomorphisms $\delta : P \to L, \delta^- : P^- \to L$ onto a finite group L such that $\delta\,|_{P\cap P^-} = \delta^-\,|_{P\cap P^-}$. We can then construct a finite three $\mathcal{J}$-class monoid $M = M(G, P, P^-, L) = G \sqcup J \sqcup \{0\}$. Here $J = (G \times L \times G)/\equiv$ with $s_1 \equiv s_2$ defined for $s_1 = (x_1, l_1, y_1), s_2 = (x_2, l_2, y_2) \in G \times L \times G$ by the conditions $x_2^{-1}x_1 \in P, y_2y_1^{-1} \in P^-$ and $\delta(x_2^{-1}x_1)l_1 = l_2\delta^-(y_2y_1^{-1})$. For $s = (x, l, y), \overline{s} = (\overline{x}, \overline{l}, \overline{y}) \in J, g \in G$ the multiplication rule is given by:

$$s\overline{s} = \begin{cases} (x, l\delta^-(p)\delta(q)\overline{l}, \overline{y}) & \text{if } y\overline{x} = pq, p \in P^-, q \in P, \\ 0 & \text{if } y\overline{x} \notin P^-P \end{cases}$$

$$sg = (x, l, yg), \quad gs = (gx, l, y).$$

Monoids of this type were defined and investigated by Putcha, cf. [8]. Consider the coset decompositions $G = \bigsqcup_{j=1}^{n} P^-a_j = \bigsqcup_{i=1}^{m} b_iP, a_1 = 1, b_1 = 1$. Every element of J is uniquely expressible in the form (b_i, l, a_j) where b_i, a_j are as above and $l \in L$. If e denotes the idempotent $(1, 1, 1)$, then $J = GeG$. We say that $s, \overline{s} \in M$ are conjugate if $gsg^{-1} = \overline{s}$ for some $g \in G$. It is easy to see that any two idempotents in J are conjugate. In some sense the converse is also true, cf. [8]. J is a $\mathcal{J}$-class of M and J^0 is a completely 0-simple semigroup. J^0 has a Rees presentation $\mathcal{M}(L, I_1, I_2, \mathcal{P})$ where $I_1 = \{b_1, b_2, \ldots, b_m\}, I_2 = \{a_1, a_2, \ldots, a_n\}$ and the sandwich matrix $\mathcal{P} = (p_{j,i})$ is defined by

$$p_{j,i} = \begin{cases} \delta^-(p)\delta(q) & \text{if } a_jb_i = pq, p \in P^-, q \in P, \\ 0 & \text{if } a_jb_i \notin P^-P. \end{cases}$$

Note that $\mathcal{P}$ depends on the choice of coset representatives.

Let K be a field. By a representation of M we will mean a homomorphism $\rho : M \to M_n(K)$ such that $\rho(1) = 1$ and $\rho(0) = 0$. Let $K_0[M] = K[M]/K0_M$ denote the contracted semigroup algebra of M. Then the representations of M are in a natural one-to-one correspondence with left $K_0[M]$-modules.

Let X be an irreducible $K[L]$-module. Let V be the linear space over K with basis $b_1, b_2, \ldots, b_m$. Define $\overline{X} = V \otimes X$, where the tensor product is over K. We may give $\overline{X}$ the structure of a $K[M]$-module as follows :

$$s(b_i \otimes x) = b_{s(i)} \otimes \gamma(s, i)x \text{ for } s \in M, x \in X$$

where $\gamma(s, i)$, and $s(i)$ if $\gamma(s, i) \neq 0$, are defined by

$$s(b_i, 1, 1) = (b_{s(i)}, \gamma(s, i), 1) \text{ with } \gamma(s, i) \in L \cup \{0\}$$

(every $(x,0,y)$ will be identified with zero of M, so $s(i)$ need not be defined for $\gamma(s,i)=0$). That the action is well-defined is obvious. We must show that

$$s_1[s_2(b_i \otimes x)] = (s_1s_2)(b_i \otimes x) \text{ for } s_i \in M, x \in X. \tag{1}$$

First, note that $s_2(b_i,1,1) = (b_{s_2(i)}, \gamma(s_2,i),1)$. Thus, if $0 \neq \gamma(s_2,i) = \delta^-(q)$ for some $q \in P^-$, we have

$$(s_1s_2)(b_i,1,1) = s_1(b_{s_2(i)}, \gamma(s_2,i),1) = s_1(b_{s_2(i)}, \delta^-(q),1)$$

$$= s_1(b_{s_2(i)},1,1)q = (b_{s_1(s_2(i))}, \gamma(s_1,s_2(i)),1)q = (b_{s_1(s_2(i))}, \gamma(s_1,s_2(i))\delta^-(q),1)$$
$$= (b_{s_1(s_2(i))}, \gamma(s_1,s_2(i))\gamma(s_2,i),1).$$

The equality

$$(s_1s_2)(b_i,1,1) = (b_{s_1(s_2(i))}, \gamma(s_1,s_2(i))\gamma(s_2,i),1)$$

also holds if $\gamma(s_2,i)=0$. Furthermore $(s_1s_2)(b_i,1,1) = (b_{(s_1s_2)(i)}, \gamma(s_1s_2,i),1)$. This implies that

$$s_1(s_2(i)) = (s_1s_2)(i) \text{ if } \gamma(s_1s_2,i) \neq 0,$$
$$\gamma(s_1,s_2(i))\gamma(s_2,i) = \gamma(s_1s_2,i).$$

Therefore (1) follows :

$$s_1[s_2(b_i \otimes x)] = s_1(b_{s_2(i)} \otimes \gamma(s_2,i)x) = b_{s_1(s_2(i))} \otimes \gamma(s_1,s_2(i))\gamma(s_2,i)x$$
$$= b_{(s_1s_2)(i)} \otimes \gamma(s_1s_2,i)x = (s_1s_2)(b_i \otimes x).$$

Note that $\overline{X}$ is related to Schutzenberger representations, [2, Section 3.5]. Our aim is to describe irreducible representations of M explicitly in the case where $K_0[M]$ is semisimple. The general theory of representations of semigroups, [2, chapter 5], cannot be applied since this requires the exact form of the inverse of $\mathcal{P}$ over $K[L]$. $\mathcal{P}^{-1}$ is known in some special cases only, cf. [9].

With the above notation we can state our first result.

Theorem 2.1. *Assume that K is an algebraically closed field of characteristic zero and $|P^-| \geq |P|$. Then the monoid algebra $K_0[M]$ is semisimple if and only if all $K[M]$-modules $\overline{X}$ (for all irreducible $K[L]$-modules X) are irreducible. Moreover, in this case all irreducible $K[M]$-modules with e acting not as 0 are of this type.*

Proof. Suppose first that all $K[M]$-modules $\overline{X}$ are irreducible. Let $W_1, W_2, \ldots, W_r$ ($V_1, V_2, \ldots, V_s$ respectively) be representatives of isomorphism classes of irreducible modules over $K[G]$ ($K[L]$). Then $W_1, W_2, \ldots, W_r$ are also $K[M]$-modules with $K[J]$ acting as zero. We will prove that $K[M]$-modules $W_1, W_2, \ldots, W_r, \overline{V}_1, \overline{V}_2, \ldots, \overline{V}_s$ are pairwise nonisomorphic. It is sufficient to show that $\overline{V}_i \not\approx \overline{V}_j$ for $i \neq j$. First, note that

$$e(b_i,1,1) = \begin{cases} (b_1, \delta^-(p^-)\delta(p),1) & \text{if } b_i = p^-p \text{ for some } p^- \in P^-, p \in P, \\ 0 & \text{otherwise.} \end{cases}$$

This gives $e(1)=1$ if $\gamma(e,i) \neq 0$. Hence $e(b_i \otimes x) = b_1 \otimes \gamma(e,i)x$ for $x \in V_k$ and then $e\overline{V}_k \subseteq b_1 \otimes V_k$. Since $e(b_1 \otimes x) = b_1 \otimes x$ for $x \in V_k$, we have $e\overline{V}_k = b_1 \otimes V_k$. Let us define $\varphi : e\overline{V}_k = b_1 \otimes V_k \to V_k$ by $\varphi(b_1 \otimes v) = v, v \in V_k$. Then $e\overline{V}_k$ is a $K[(1,L,1)] = K[L]$-module and φ is an isomorphism of $K[L]$-modules. If $K[M]$-modules $\overline{V}_i, \overline{V}_j$ are isomorphic, then $K[L]$-modules $e\overline{V}_i, e\overline{V}_j$ are isomorphic. Hence $K[L]$-modules V_i, V_j are isomorphic and $i=j$. Since $K[G]$ is semisimple,

$\sum_{i=1}^{r}(\dim_K W_i)^2 = |G|$. We continue in this fashion, obtaining $\sum_{i=1}^{s}(\dim_K \overline{V}_i)^2 = \sum_{i=1}^{s}(|G/P|\dim_K V_i)^2 = |G/P|^2|L|$ since $K[L]$ is semisimple. Hence

$$\sum_{i=1}^{r}(\dim_K W_i)^2 + \sum_{i=1}^{s}(\dim_K \overline{V}_i)^2 \geq |G| + |G/P||L||G/P^-| = \dim_K K_0[M].$$

This proves that $K_0[M]$ is semisimple ($|P| = |P^-|$ in particular) and all irreducible $K[M]$-modules are among $W_1, W_2, \dots, W_r, \overline{V}_1, \overline{V}_2, \dots, \overline{V}_s$.

Conversely, assume that $K_0[M]$ is semisimple. Let X be an irreducible $K[L]$-module. Define $\overline{X}_0 = \{v \in \overline{X} : (GeG)v = 0\}$. $\overline{X}_0$ is a $K[M]$-submodule of $\overline{X}$. Since $K_0[M]$ is semisimple there exists a $K[M]$-submodule $\overline{X}_1$ of $\overline{X}$ such that $\overline{X}_0 \oplus \overline{X}_1 = \overline{X}$. Then $e\overline{X} = e\overline{X}_0 + e\overline{X}_1 = e\overline{X}_1 \subseteq \overline{X}_1$. Thus $b_1 \otimes X \subseteq \overline{X}_1$. This gives $K[G](b_1 \otimes X) \subseteq \overline{X}_1$. Since also $K[G](b_1 \otimes x) = \overline{X}$, we have $\overline{X}_1 = \overline{X}$ and so $\overline{X}_0 = 0$. Let $v \in \overline{X} \setminus \{0\}$. Then there exists $g \in G$ such that $egv \neq 0$ since $\overline{X}_0 = 0$. Because $egv \in b_1 \otimes X$ and $K[L]$-module X is irreducible we have $K[(1, L, 1)]egv = b_1 \otimes X$. Then $K[M]egv \supseteq K[G]b_1 \otimes X = \overline{X}$. Hence $K[M]v = \overline{X}$. We have shown that $\overline{X}$ is an irreducible $K[M]$-module. This proves the assertion. □

Note that the above theorem may be easily generalised to the case of arbitrary monoids of Lie type.

The next question is to decide when $\overline{X}$ is an irreducible $K[M]$-module. We shall see that the answer can be given in terms of group representation theory. First we need some preparatory observations.

It is easy to see, that $K[G]$-module $\overline{X}$ is isomorphic to the induced module $Ind_P^G(X)$, where $K[P]$-module structure on X comes from $\delta : P \to L$. Define $\xi : \overline{X} \to K[G/P] \otimes_{K[P]} X = Ind_P^G(X)$ by $\xi(b_i \otimes x) = b_i \otimes x$ for $x \in X$ and extending by linearity. First we will show that ξ is a homomorphism of $K[G]$-modules. Let $g \in G, x \in X$. Then $g(b_i, 1, 1) = (b_{g(i)}, \delta(p(g, i)), 1)$ where $p(g, i) \in P$ is defined by $gb_i = b_{g(i)}p(g, i)$. Hence $g(b_i \otimes x) = b_{g(i)} \otimes \delta(p(g, i))x$. This gives $\xi(g(b_i \otimes x)) = \xi(b_{g(i)} \otimes \delta(p(g, i))x) = b_{g(i)} \otimes \delta(p(g, i))x$. Since also $g\xi(b_i \otimes x) = (gb_i) \otimes x = (b_{g(i)}p(g, i)) \otimes x = b_{g(i)} \otimes (p(g, i)x) = b_{g(i)} \otimes \delta(p(g, i))x$ then $\xi(g(b_i \otimes x)) = g\xi(b_i \otimes x)$. Since $Ind_P^G X = \bigoplus_i b_i \otimes X$, ξ is surjective. But $\dim_K \overline{X} = \dim_K Ind_P^G X$, so ξ is an isomorphism. We can consider also $Ind_{P^-}^G X$, where $K[P^-]$-module structure on X comes from $\delta^- : P^- \to L$. Let us define the homomorphism of $K[G]$-modules

$$\Phi = Ind_{P,P^-}^G(id_X) : Ind_{P^-}^G X \to Ind_P^G X$$

by

$$\Phi(y) = \sum_{i=1}^{r} b_i \Phi_1(b_i^{-1} y) \text{ for } y \in Ind_{P^-}^G X$$

where $\Phi_1 : Ind_{P^-}^G X \to X \subseteq Ind_P^G X$ is given by:

$$\Phi_1(px) = p\, id_X(x) \text{ for } p \in P, x \in X,$$

$$\Phi_1(gx) = 0 \text{ if } g \in G \setminus PP^-, x \in X.$$

First we must show that Φ_1 is well-defined. Assume that equality $p_1x_1 = p_2x_2$ holds in $Ind_{P^-}^G X$ for some $p_1, p_2 \in P$ and $x_1, x_2 \in X$. Then there exists $p^- \in P^-$ such that $p_1 = p_2p^-$ and $x_2 = \delta^-(p^-)x_1$. Hence $p_2^{-1}p_1 = p^- \in P \cap P^-$. The homomorphisms δ and δ^- agree on $P \cap P^-$, so $\delta(p_2)^{-1}\delta(p_1) = \delta^-(p^-)$. Thus in $Ind_P^G X$ we have: $p_1x_1 = \delta(p_1)x_1 = \delta(p_2)\delta^-(p^-)x_1 = \delta(p_2)x_2 = p_2x_2$. It is easy to see that Φ_1 is a homomorphism of $K[P]$-modules. If $g \in G$, then $g^{-1}b_i = b_{g(i)}p(g^{-1},i)$ for some $p(g^{-1},i) \in P$. Let $y \in Ind_{P^-}^G$. Then

$$\Phi(gy) = \sum_{i=1}^r b_i\Phi_1(b_i^{-1}gy) = \sum_{i=1}^r b_i\Phi_1(p^{-1}(g^{-1},i)b_{g(i)}^{-1}y)$$

$$= \sum_{i=1}^r b_i p^{-1}(g^{-1},i)\Phi_1(b_{g(i)}^{-1}y) = \sum_{i=1}^r g b_{g(i)}\Phi_1(b_{g(i)}^{-1}y) = g\sum_{i=1}^r b_i\Phi_1(b_i^{-1}y) = g\Phi(y).$$

This shows that Φ is indeed a homomorphism of $K[G]$-modules. With the above notation we are now able to characterize the irreducibility of $\overline{X}$.

Theorem 2.2. *Assume that K is an algebraically closed field of characteristic zero and $|P^-| \geq |P|$. Then $\overline{X}$ is an irreducible $K[M]$-module if and only if $Ind_{P,P^-}^G(id_X)$ is an isomorphism.*

Proof. Let $\langle\ |\ \rangle : X \times X \to K$ be an L-invariant, nonsingular bilinear form on K-linear space X. Define $\langle\ |\ \rangle : \overline{X} \times \overline{X} \to K$ by $\langle \sum_{i=1}^r b_i \otimes x_i | \sum_{i=1}^r b_i \otimes \overline{x}_i\rangle = \sum_{i=1}^r \langle x_i|\overline{x}_i\rangle$. Then $\langle\ |\ \rangle$ is a G-invariant nonsingular bilinear form on $\overline{X}$. Let $\ker(e) = \{v \in \overline{X} : ev = 0\}$. We claim that $\ker e \perp \xi^{-1}\Phi(X)$. Let $\sum_{i=1}^r b_i \otimes x_i \in \ker(e), x_i \in X$. Since $e(b_i,1,1) = (b_1,\gamma(e,i),1)$, then $0 = e\sum_{i=1}^r b_i \otimes x_i = \sum_{i=1}^r b_1 \otimes \gamma(e,i)x_i = b_1 \otimes \sum_{i=1}^r \gamma(e,i)x_i$. So we have $\sum_{i=1}^r \gamma(e,i)x_i = 0$ if and only if $\sum_{i=1}^r b_i \otimes x_i \in \ker(e)$.

Next, we will describe $\Phi(X)$. Let $x \in X$. Then by the definition $\Phi(x) = \sum_{i=1}^r b_i\Phi_1(b_i^{-1}x)$. If $b_i^{-1} = p_i^{-1}(p_i^-)^{-1}$, $p_i \in P, p_i^- \in P^-$, then

$$b_i\Phi_1(b_i^{-1}x) = b_i\Phi_1(p_i^{-1}(\delta^-((p_i^-)^{-1})x)) = b_i\delta(p_i^{-1})\delta^-((p_i^-)^{-1})x$$

$$= b_i[\delta^-(p_i^-)\delta(p_i)]^{-1}x = b_i\gamma(e,i)^{-1}x.$$

On the other hand $b_i\Phi_1(b_i^{-1}x) = 0$ if $b_i \notin P^-P$. Hence

$$\Phi(x) = \sum_{i:\gamma(e,i)\neq 0} b_i\gamma(e,i)^{-1}x.$$

This implies that

$$\xi^{-1}\Phi(x) = \sum_{i:\gamma(e,i)\neq 0} b_i \otimes \gamma(e,i)^{-1}x.$$

Let $\sum_{i=1}^r b_i \otimes x_i \in \ker(e), x_i, x \in X$. Then

$$\langle\sum_{i=1}^r b_i \otimes x_i|\xi^{-1}\Phi(x)\rangle = \langle\sum_{i=1}^r b_i \otimes x_i| \sum_{i:\gamma(e,i)\neq 0} b_i \otimes \gamma(e,i)^{-1}x\rangle$$

$$= \sum_{i:\gamma(e,i)\neq 0}\langle x_i|\gamma(e,i)^{-1}x\rangle = \sum_{i:\gamma(e,i)\neq 0}\langle\gamma(e,i)x_i|x\rangle = \langle\sum_{i=1}^r\gamma(e,i)x_i|x\rangle = \langle 0|x\rangle = 0.$$

Since X is an irreducible $K[L]$-module, $\xi^{-1}\Phi(X)$ is $K[L]$-isomorphic to X or 0. If $\xi^{-1}\Phi(X) = 0$ then $\Phi(gX) = 0$ for all $g \in G$, so $\Phi = 0$, a contradiction. This proves that the former holds. Hence $\dim_K \xi^{-1}\Phi(x) = \dim_K X$. Since $e\overline{X}$ and X are $K[L]$-isomorphic, we have $\dim_K \ker(e) = \dim_K \overline{X} - \dim_K X$. Using the fact that $\xi^{-1}\Phi(X) \subseteq (\ker e)^\perp$ and $\dim_K \xi^{-1}\Phi(X) + \dim_K \ker(e) = \dim_K \overline{X}$ we get $\xi^{-1}\Phi(X) = (\ker e)^\perp$.

Now we are ready to prove the assertion of the theorem. Assume first that $\overline{X}$ is an irreducible $K[M]$-module. Let $v_0 \in (\bigcap_{g\in G} g\ker(e)) \setminus \{0\}$. Then $egv_0 = 0$ for all $g \in G$. Hence $GeGv_0 = 0$. Define $Y = \{v \in \overline{X} : GeGv = 0\}$. Y is a $K[M]$-submodule of $\overline{X}$ since $GeG \cup \{0\} = J \cup \{0\}$ is an ideal of M. Hence $Y = \overline{X}$, so $e\overline{X} = 0$. This contradiction proves that $\bigcap_{g\in G} g\ker(e) = 0$. This implies that $\sum_{g\in G} g(\ker e)^{\perp} = \overline{X}$. Hence $\overline{X} = \sum_{g\in G} g\xi^{-1}\Phi(X) = \xi^{-1}\Phi(\sum_{g\in G} gX) = \xi^{-1}\Phi(Ind_{P^-}^G X)$ and Φ is surjective. Since also $\dim_K Ind_{P^-}^G X \leq \dim_K Ind_P^G X$, Φ is an isomorphism ($|P| = |P^-|$ in particular). Conversely, suppose that Φ is an isomorphism. Thus $\overline{X} = \xi^{-1}\Phi(Ind_{P^-}^G X) = \xi^{-1}\Phi(\sum_{g\in G} gX) = \sum_{g\in G} g\xi^{-1}\Phi(X)$. Hence $\bigcap_{g\in G} g\ker(e) = 0$. Let $v \in \overline{X} \setminus \{0\}$. Then there exists $g \in G$ with $egv \neq 0$. As at end of the proof of Theorem 2.1 this implies that the $K[M]$-module $\overline{X}$ is irreducible. This completes the proof.

3. Semisimplicity

Let G be a finite group. Then G admits a BN-pair if there are subgroups B, N of G which generate G such that $T = B \cap N \trianglelefteq N$ and the Weyl group $W = N/T$ has a generating set of elements s_i, $i \in I$, with $s_i^2 = 1$ and

(i) $s_iBw \subseteq Bs_iwB \cup BwB$ for every s_i and $w \in W$,

(ii) $s_iBs_i \neq B$ for every s_i.

Then W is a Coxeter group. If $w \in W$, we define $l(w)$ to be the minimal length of w as a product of the generators $s_i, i \in I$. In particular $l(w) = 0$ if and only if $w = 1$. The conjugates of B are called Borel subgroups. Any subgroup P of G containing a Borel subgroup is called a parabolic subgroup of G. Let W_J be the subgroup of W generated by elements s_i with $i \in J$ for some $J \subseteq I$. Then P_J denotes the standard parabolic subgroup BW_JB. Any parabolic subgroup P of G is conjugate to a unique P_J. It turns out that W_J has a unique element $(w_0)_J$ of maximal length. Then $(w_0)_J$ is of order 2. Write $(w_0)_I = w_0$ and $B^- = w_0Bw_0$. Then $P_J^- = B^-W_JB^-$ is a parabolic subgroup of G which is called opposite to P_J. We will assume as in [1, chapter 2], that G admits a split B-N pair satisfying some commutator relations. Then there exists a normal subgroup U of B such that $B = UT, U \cap T = \{1\}$. Let Φ be the root system of W and $\Delta = \{\alpha_1, \alpha_2, \dots, \alpha_l\} \subseteq \Phi$ ($l = |I|$) the set of simple roots. Each root $\alpha \in \Phi$ has the form $\alpha = \sum_{i=1}^{l} \lambda_i\alpha_i$ where either all $\lambda_i \geq 0$ or all $\lambda_i \leq 0$. Roots with all $\lambda_i \geq 0$ are called positive roots ($\alpha > 0$) and their set is denoted by Φ^+. Let $\Delta_J = \{\alpha_i : i \in J\}$ ($J \subseteq I$). Define $\Phi_J = W_J(\Delta_J)$. Then Φ_J is a root system for W_J with the set of simple roots Δ_J . We also define subgroups of G: $U^- = U^{w_0}, X_i = U \cap U^{w_0 s_i}, U_w = U \cap U^{w_0 w}$ for $w \in W$. If $w(\alpha_i) = \alpha, w \in W, \alpha \in \Phi$, we will denote the root subgroup wX_iw^{-1} by X_α. The definition is independent of the choice of w and α_i. Then U_w is a product of X_β such that $\beta > 0$ and $w(\beta) < 0$. Let $U_J = U \cap U^{(w_0)_J}, U_J^- = U \cap U^{w_0(w_0)_J}, L_J = \langle T, X_\alpha : \alpha \in \Phi_J\rangle$ $(= P_J \cap P_J^-)$. Then U_J, U_J^- are normal subgroups of P_J, P_J^- respectively and we have the Levi decompositions $P_J = U_JL_J$, $P_J^- = U_J^-L_J$ with $U_J \cap L_J = \{1\}$, $U_J^- \cap L_J = \{1\}$. L_J is called a Levi factor of P_J. Any Levi factor of P_J is conjugate to L_J by some element of U_J. U_J, U_J^- are called unipotent radicals of P_J, P_J^- respectively. U_J is the product of X_β with $\beta > 0$ and $\beta \notin \Phi_J$ in any order. Let $J_1, J_2 \subseteq I$. We define $D_{J_1,J_2} = \{w \in W : w^{-1}(\Delta_{J_1}) > 0 \text{ and } w(\Delta_{J_2}) > 0\}$. Then $W = W_{J_1}D_{J_1,J_2}W_{J_2}$ and any $w \in D_{J_1,J_2}$ is the unique element of minimal length in $W_{J_1}wW_{J_2}$. Moreover we have $G = \bigsqcup_{w\in D_{J_1,J_2}} P_{J_1}wP_{J_2}$. See [1, chapter 2], for details.

Next we restrict our attention to monoids $M = M(G, P, P^-, L)$ where G is a finite group with a split BN-pair satisfying some commutator relations and P, P^- are parabolic subgroups of G with a common Levi factor L. Then $P = UL, P^- = U^-L$ where U, U^- are the unipotent radicals of P, P^- respectively. Homomorphisms δ and δ^- are defined by $\delta(ul) = l, \delta^-(u^-l) = l$ for $l \in L, u \in U, u^- \in U^-$. The monoid defined by $(gPg^{-1}, gLg^{-1}, gP^-g^{-1})$, $g \in G$, is isomorphic to that defined by (P, L, P^-). Let us write $(P, L, P^-) \sim (gPg^{-1}, gLg^{-1}, gP^-g^{-1})$ in this case. We will find $g \in G$ such that $(gPg^{-1}, gLg^{-1}, gP^-g^{-1})$ is 'standard'. There exist $g_1, g_2 \in G$ and $J_1, J_2 \subseteq I$ with $P = P_{J_1}^{g_1}, P^- = P_{J_2}^{g_2}$. Since $G = P_{J_1}D_{J_1,J_2}P_{J_2}$, then $g_1g_2^{-1} = p_1wp_2$ where $p_i \in P_{J_i}, w \in D_{J_1,J_2}$. Thus

$$(P, L, P^-) \sim (P_{J_1}, L^{g_1^{-1}}, P_{J_2}^{g_2g_1^{-1}}) \sim (P_{J_1}, L^{g_1^{-1}}, P_{J_2}^{p_2^{-1}w^{-1}p_1^{-1}})$$

$$\sim (P_{J_1}^{p_1}, L^{g_1^{-1}p_1}, (P_{J_2}^{p_2^{-1}})^{w^{-1}}) = (P_{J_1}, \overline{L}, P_{J_2}^{w^{-1}}).$$

So we can assume that $P = P_{J_1}, P^- = P_{J_2}^{w^{-1}}, w \in D_{J_1,J_2}$. Next $L = u_1L_{J_1}u_1^{-1} = (u_2L_{J_2}u_2^{-1})^{w^{-1}}$ for some $u_i \in U_{J_i}$. By [1, Prop.2.8.3] $P_{J_1} \cap P_{J_2}^{w^{-1}} \subseteq P_K$ where $\Delta_K = \Delta_{J_1} \cap w(\Delta_{J_2})$. Hence $u_1L_{J_1}u_1^{-1} \subseteq P_K$. Since $U_{J_1} \subseteq U_K$, then $L_{J_1} \subseteq P_K$. This gives $P_{J_1} = U_{J_1}L_{J_1} \subseteq P_K$. So $J_1 \subseteq K$ and $\Delta_{J_1} \subseteq w(\Delta_{J_2})$ by the definition of K . Similar considerations applied to $P_{J_2} \cap P_{J_1}^{w}$ $(w^{-1} \in D_{J_2,J_1})$ allow us to prove that $\Delta_{J_2} \subseteq w^{-1}(\Delta_{J_1})$. Hence $\Delta_{J_1} = w(\Delta_{J_2})$. By [1, Th.2.8.7]

$$P_{J_1} \cap P_{J_2}^{w^{-1}} = (U_{J_1} \cap U_{J_2}^{w^{-1}})(U_{J_1} \cap L_{J_2}^{w^{-1}})(L_{J_1} \cap U_{J_2}^{w^{-1}})L_K$$

where $\Delta_K = \Delta_{J_1} \cap w(\Delta_{J_2})$. Since $\Delta_{J_1} = w(\Delta_{J_2})$, we have $K = J_1$. By [1, Cor.2.8.8] $U_{J_1} \cap L_{J_2}^{w^{-1}} = \{1\}$ and $L_{J_1} \cap U_{J_2}^{w^{-1}} = \{1\}$. Hence $P_{J_1} \cap P_{J_2}^{w^{-1}} = (U_{J_1} \cap U_{J_2}^{w^{-1}})L_{J_1} = (U_{J_1} \cap U_{J_2}^{w^{-1}})L$. This proves that δ and δ^- agree on $P_{J_1} \cap P_{J_2}^{w^{-1}}$.

Next we will show that $U_{J_2}^{w^{-1}} = (U_{J_1} \cap U_{J_2}^{w^{-1}})(U_{J_1}^- \cap U_{J_2}^{w^{-1}})$. Since $U_{J_2}^{w^{-1}}$ is a product of $X_\alpha^{w^{-1}}$, where $\alpha > 0$ and $\alpha \notin \Phi_{J_2}$, in any order, it is sufficient to show that $X_\alpha^{w^{-1}} \subseteq U_{J_1} \cap U_{J_2}^{w^{-1}}$ or $X_\alpha^{w^{-1}} \subseteq U_{J_1}^- \cap U_{J_2}^{w^{-1}}$. We see that $w(\alpha) \notin w(\Phi_{J_2}) = \Phi_{J_1}$. Hence, if $w(\alpha) > 0$, then $X_\alpha^{w^{-1}} = X_{w(\alpha)} \subseteq U_{J_1} \cap U_{J_2}^{w^{-1}}$ and, if $w(\alpha) < 0$, then $X_{w(\alpha)} \subseteq U_{J_1}^- \cap U_{J_2}^{w^{-1}}$. So we can factorize $u_2^{w^{-1}}$ as $u_2^{w^{-1}} = \overline{u}_1\overline{\overline{u}}_1$ where $\overline{u}_1 \in U_{J_1} \cap U_{J_2}^{w^{-1}}, \overline{\overline{u}}_1 \in U_{J_1}^- \cap U_{J_2}^{w^{-1}}$. Thus

$$\overline{u}_1^{-1}u_1L_{J_1}u_1^{-1}\overline{u}_1 = \overline{\overline{u}}_1L_{J_1}\overline{\overline{u}}_1^{-1} \subseteq P_{J_1} \cap P_{J_1}^- = L_{J_1}.$$

So $L_{J_1} = \overline{u}_1^{-1}u_1L_{J_1}u_1^{-1}\overline{u}_1$. This gives $L = \overline{u}_1L_{J_1}\overline{u}_1^{-1}$. Here $\overline{u}_1 \in U_{J_1} \cap U_{J_2}^{w^{-1}}$. Now we have

$$(P_{J_1}, L, P_{J_2}^{w^{-1}}) \sim (P_{J_1}, L_{J_1}^{\overline{u}_1^{-1}}, P_{J_2}^{w^{-1}}) \sim (P_{J_1}^{\overline{u}_1}, L_{J_1}, P_{J_2}^{w^{-1}\overline{u}_1}) = (P_{J_1}, L_{J_1}, P_{J_2}^{w^{-1}}).$$

So there is no loss of generality in assuming that $P = P_{J_1}, L = L_{J_1}, P^- = P_{J_2}^{w^{-1}}$ and $w(\Delta_{J_2}) = \Delta_{J_1}$(this implies that $w \in D_{J_1,J_2}$).

Let K_1 satisfy $K_1 \subseteq J_1$. K_2 is defined by $w(\Delta_{K_2}) = \Delta_{K_1}$. Clearly $K_2 \subseteq J_2$. Let Y be a $K[L_{K_1}]$-module. Consider the homomorphism of $K[G]$-modules

$$\phi = Ind_{P_{K_1}, P_{K_2}^{w^{-1}}}^G(id_Y) : Ind_{P_{K_2}^{w^{-1}}}^G Y \to Ind_{P_{K_1}}^G Y$$

defined as before Theorem 2.2 ($K[P_{K_2}^{w^{-1}}]$-structure on Y is determined by the trivial action of $U_{K_2}^{w^{-1}}$ and similarly $K[P_{K_1}]$-structure by U_{K_1} acting trivially). $P_{K_1} \cap L_{J_1}$

is a standard parabolic subgroup of L_{J_1} and we have the Levi decomposition $P_{K_1} \cap L_{J_1} = (U_{K_1} \cap L_{J_1})L_{K_1}$, cf.[1, Prop.2.8.9]. Similarly we define

$$\overline{\phi} = Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1},(P_{K_2}\cap L_{J_2})^{w^{-1}}}(id_Y) : Ind^{L^{w^{-1}}_{J_2}}_{(P_{K_2}\cap L_{J_2})^{w^{-1}}} Y \to Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1}} Y$$

(here $(U_{K_2} \cap L_{J_2})^{w^{-1}}, U_{K_1} \cap L_{J_1}$ act trivially on suitable copies of Y). Finally, we define

$$Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}) : Ind^G_{P^{w^{-1}}_{J_2}}(Ind^{L^{w^{-1}}_{J_2}}_{(P_{K_2}\cap L_{J_2})^{w^{-1}}} Y) \to Ind^G_{P_{J_1}}(Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1}} Y)$$

(where $U^{w^{-1}}_{J_2}$ acts as 1 on $Ind^{L^{w^{-1}}_{J_2}}_{(P_{K_2}\cap L_{J_2})^{w^{-1}}} Y$, U_{J_1} acts as 1 on $Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1}} Y$). Notice that $Ind^G_{P_{J_1}} Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1}} Y = Ind^G_{P_{K_1}} Y$. The following lemma is an analogue of this equality for Ind homomorphisms.

Lemma 3.1. *With the above notation we have*

$$Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y) = Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1},(P_{K_2}\cap L_{J_2})^{w^{-1}}}(id_Y)).$$

Proof. By the definition of homomorphisms of type Ind we have

$$(Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1(p_{J_1}\overline{y}) = p_{J_1}\overline{\phi}(\overline{y}) \text{ for } p_{J_1} \in P_{J_1}, \overline{y} \in Ind^{L^{w^{-1}}_{J_2}}_{(P_{K_2}\cap L_{J_2})^{w^{-1}}} Y.$$

Let $p_{K_1} \in P_{K_1}$. Then $p_{K_1} = u_{J_1} l_{J_1}$ for some $u_{J_1} \in U_{J_1}, l_{J_1} \in L_{J_1}$. Since $u_{J_1} \in U_{K_1}$, we have $l_{J_1} \in P_{K_1} \cap L_{J_1}$. Moreover, let $y \in Y$. Then

$$((Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1)_1(p_{K_1}y) = (p_{K_1}\overline{\phi}(y))_1 = (u_{J_1}l_{J_1}\overline{\phi}(y))_1 = (l_{J_1}\overline{\phi}(y))_1$$

$$= (\overline{\phi}(l_{J_1}y))_1 = \overline{\phi}_1(l_{J_1}y) = l_{J_1}y = u_{J_1}l_{J_1}y = p_{K_1}y = (Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y))_1(p_{K_1}y).$$

Next we prove that $((Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1)_1(gy) = 0 = (Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y))_1(gy)$ for $g \notin P_{K_1}P^{w^{-1}}_{K_2}$ and $y \in Y$. The second equality follows from the definition. The first equality holds if $g \notin P_{J_1}P^{w^{-1}}_{J_2}$ since $(Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1(gy) = 0$. So we can assume that $g = p_{J_1}p^{w^{-1}}_{J_2}$ for some $p_{J_i} \in P_{J_i}$. Moreover, let $p_{J_1} = u_{J_1}l_{J_1}$, $p_{J_2} = u_{J_2}l_{J_2}$ where $u_{J_i} \in U_{J_i}, l_{J_i} \in L_{J_i}$. Then we have

$$((Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1)_1(gy) = [(Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1(p_{J_1}u^{w^{-1}}_{J_2}(l^{w^{-1}}_{J_2}y))]_1$$

$$= [(Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1(p_{J_1}(l^{w^{-1}}_{J_2}y))]_1 = (p_{J_1}\overline{\phi}(l^{w^{-1}}_{J_2}y))_1 = (u_{J_1}l_{J_1}\overline{\phi}(l^{w^{-1}}_{J_2}y))_1$$

$$= (l_{J_1}\overline{\phi}(l^{w^{-1}}_{J_2}y))_1 = (\overline{\phi}(l_{J_1}l^{w^{-1}}_{J_2}y))_1 = \overline{\phi}_1(l_{J_1}l^{w^{-1}}_{J_2}y).$$

Since $u_{J_1}l_{J_1}u^{w^{-1}}_{J_2}l^{w^{-1}}_{J_2} \notin P_{K_1}P^{w^{-1}}_{K_2}$ by the assumption, then

$$l_{J_1}l^{w^{-1}}_{J_2} \notin (u^{-1}_{J_1}P_{K_1})(P_{K_2}(u^{-1}_{J_2})^{l_{J_2}})^{w^{-1}} = P_{K_1}P^{w^{-1}}_{K_2}.$$

This implies $l_{J_1}l^{w^{-1}}_{J_2} \notin (P_{K_1} \cap L_{J_1})(P_{K_2} \cap L_{J_2})^{w^{-1}}$. Hence

$$((Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1)_1(gy) = \overline{\phi}_1(l_{J_1}l^{w^{-1}}_{J_2}y) = 0.$$

We have proved that

$$((Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\phi}))_1)_1(gy) = (Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y))_1(gy)$$

for $y \in Y$, $g \in P_{K_1}$ and also for $g \notin P_{K_1} P_{K_2}^{w^{-1}}$. So $((Ind^G_{P_{J_1}, P_{J_2}^{w^{-1}}}(\overline{\phi}))_1)_1 = (Ind^G_{P_{K_1}, P_{K_2}^{w^{-1}}})_1$. This establishes the desired formula.

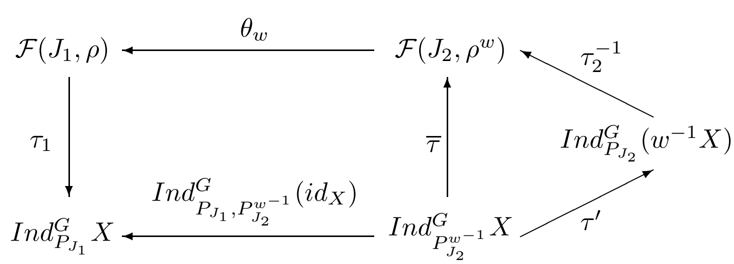
We are now ready for the main result of this section.

Theorem 3.2. *Assume that $M = M(G, P, P^-, L)$ where G is a finite group with a split BN-pair satisfying some commutator relations and P, P^- are parabolic subgroups of G with a common Levi factor L. Let K be an algebraically closed field of characteristic zero. Then the semigroup algebra $K_0[M]$ is semisimple.*

Proof. By [3] we can assume that K is the field of complex numbers. We know, that it is sufficient to prove the theorem for $P = P_{J_1}$, $L = L_{J_1}, P^- = P_{J_2}^{w^{-1}}$ where $w(\Delta_{J_2}) = \Delta_{J_1}$. By the symmetry of our problem it may be assumed that $|P^-| \geq |P|$. By Theorem 2.1 it suffices to prove that all $K[M]$-modules $\overline{X}$, where X runs over the set of all irreducible $K[L_{J_1}]$-modules, are irreducible. Fix some X. In view of Theorem 2.2 this is equivalent to $Ind^G_{P_{J_1}, P_{J_2}^{w^{-1}}}(id_X)$ being an isomorphism of $K[G]$-modules. Suppose that the representation of L_{J_1} in $End_K X$ is cuspidal. This means that the $K[L_{J_1}]$-module X is not a component of any induced module $Ind^{L_{J_1}}_{P_{K_1} \cap L_{J_1}} Y$, where $K_1 \subset J_1$, Y is a $K[K_1]$-module and $U_{K_1} \cap L_{J_1}$ acts trivially on Y. As in [1, 10.1] we will consider a functional representation of $Ind^G_{P_{J_1}} X$. Let $\rho : L_{J_1} \to End_K X$ be the associated K-representation. Consider the set $\mathcal{F}$ of all maps from G to X. $\mathcal{F}$ may be made into a left $K[G]$-module by $(gf)(x) = f(xg)$ where $x, g \in G, f \in \mathcal{F}$. Let $\mathcal{F}(J_1, \rho)$ be the subset of $\mathcal{F}$ defined by

$$\mathcal{F}(J_1, \rho) = \{f \in \mathcal{F} : f(pg) = pf(g) \text{ for } p \in P_{J_1}, g \in G\}.$$

By [1, Prop.10.1.1] $K[G]$-modules $\mathcal{F}(J_1, \rho)$ and $Ind^G_{P_{J_1}} X$ are isomorphic. Let us define a $K[G]$-homomorphism $\tau_1 : \mathcal{F}(J_1, \rho) \to Ind^G_{P_{J_1}} X$ by $\tau_1(f) = \sum_{i=1}^r g_i^{-1} f(g_i)$ where $f \in \mathcal{F}(J_1, \rho)$ and $G = \bigsqcup_{i=1}^r P_{J_1} g_i$ with $g_1 = 1$. In fact τ_1 is an isomorphism, cf. [1, proof of Prop.10.1.1]. The inverse homomorphism is determined by $\tau_1^{-1}(x) = f_x, x \in X$, where f_x satisfies $f_x(p) = px$ for $p \in P_{J_1}$ and $f_x(g) = 0$ for $g \notin P_{J_1}$. Similarly we can define a $K[G]$-isomorphism $\tau_2 : \mathcal{F}(J_2, \rho^w) \to Ind^G_{P_{J_2}}(w^{-1}X)$. We have also an obvious $K[G]$-isomorphism $\tau' : Ind^G_{P_{J_2}^{w^{-1}}} X \to Ind^G_{P_{J_2}}(w^{-1}X)$ given by $\tau'(x) = x$ for $x \in X$. Let $\overline{\tau} = \tau_2^{-1}\tau'$. Finally, consider the $K[G]$-homomorphism $\theta_w : \mathcal{F}(J_2, \rho^w) \to \mathcal{F}(J_1, \rho)$ defined by $\theta_w(f)g = |U_{J_1}|^{-1} \sum_{u \in U_{J_1}} f(w^{-1}ug)$ for any $g \in G$, see [1, Prop.10.1.3]. We get the following diagram of $K[G]$-homomorphisms.



Next we will establish a link between θ_w and $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X)$. We claim, that

$$\tau_1\theta_w\overline{\tau} = \frac{|U_{J_1}\cap U^{w^{-1}}_{J_2}|}{|U_{J_1}|} Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X). \tag{2}$$

Suppose $g \in G, x \in X$. Then we have

$$\begin{aligned}(\tau_1\theta_w\overline{\tau})_1(gx) &= [\tau_1\theta_w\overline{\tau}((gw)(w^{-1}x))]_1 = [\tau_1\theta_w(gw\overline{\tau}(w^{-1}x))]_1 \\ &= [\tau_1(\theta_w(gwf_x))]_1 = \theta_w(gwf_x)(1) = |U_{J_1}|^{-1}\sum_{u\in U_{J_1}} gwf_x(w^{-1}u) \\ &= |U_{J_1}|^{-1}\sum_{u\in U_{J_1}} f_x(w^{-1}ugw).\end{aligned}$$

If $f_x(w^{-1}ugw) \neq 0$, then $ug \in P^{w^{-1}}_{J_2}$. Hence $g \in P_{J_1}P^{w^{-1}}_{J_2}$. This gives $(\tau_1\theta_w\overline{\tau})_1(gx) = 0$ if $g \notin P_{J_1}P^{w^{-1}}_{J_2}$. Next, let $g = p_1 \in P_{J_1}$. Then $p_1 = u_1l_1$ with $u_1 \in U_{J_1}, l_1 \in L_{J_1}$. Hence $f_x(w^{-1}ugw) \neq 0$ implies that $uu_1l_1 \in P^{w^{-1}}_{J_2}$. Therefore $uu_1 \in P^{w^{-1}}_{J_2} \cap U_{J_1} = U^{w^{-1}}_{J_2} \cap U_{J_1}$, the equality following from [1, Prop.2.8.6 and Cor.2.8.8] since $w(\Delta_{J_2}) = \Delta_{J_1}$. So we have $f_x(w^{-1}ugw) = uu_1(l_1x) = l_1x$ since $uu_1 \in U^{w^{-1}}_{J_2}$. In this case $(\tau_1\theta_w\overline{\tau})_1(gx) = |U_{J_1}\cap U^{w^{-1}}_{J_2}||U_{J_1}|^{-1}l_1x$. We have shown that $(\tau_1\theta_w\overline{\tau})_1 = |U_{J_1}\cap U^{w^{-1}}_{J_2}||U_{J_1}|^{-1}[Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X)]_1$. This gives (2). According to (2), in order to show that $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X)$ is an isomorphism, it is sufficient to prove that θ_w is an isomorphism. This was done in the last part of the proof of [1, Prop.10.7.9].

Finally, assume that the representation of L_{J_1} in $End_K X$ is not cuspidal. Then $K[L_{J_1}]$-module X is a component of $Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1}} Y$ where $K_1 \subset J_1, Y$ is an irreducible $K[L_{K_1}]$-module whose associated representation is cuspidal and $U_{K_1}\cap L_{J_1}$ acts trivially on Y. We know that $\overline{\Phi} = Ind^{L_{J_1}}_{P_{K_1}\cap L_{J_1},(P_{K_2}\cap L_{J_2})^{w^{-1}}}(id_Y)$ is an isomorphism. Since $\overline{\Phi}|_{\overline{\Phi}^{-1}(X)} : \overline{\Phi}^{-1}(X) \to X$ is a component of $\overline{\Phi}$ then $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\Phi}|_{\overline{\Phi}^{-1}(X)})$ is a component of $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\Phi})$ which is equal to $Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y)$ by Lemma 3.1. The cuspidal case implies that $Ind^G_{P_{K_1},P^{w^{-1}}_{K_2}}(id_Y)$ is an isomorphism. Therefore $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\Phi}|_{\overline{\Phi}^{-1}(X)})$ is an isomorphism. But $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(\overline{\Phi}|_{\overline{\Phi}^{-1}(X)}) \simeq Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X)$, hence also $Ind^G_{P_{J_1},P^{w^{-1}}_{J_2}}(id_X)$ is an isomorphism. This completes the proof.

The above theorem yields in particular a new proof of the semisimplicity of $\mathbf{C}_0[M]$ for monoids of Lie type M, originally proved in [6].

4. Decomposing sandwich matrices

Let $S = (s_{j,i})$ be the sandwich matrix of the monoid defined by $(P_{J_1}, L_{J_1}, P^{w_1}_{J_2})$, where $w_1 \in W$, $w_1(\Delta_{J_1}) = \Delta_{J_2}$, with respect to the following coset decompositions $G = \bigsqcup_i b_iP_{J_1} = \bigsqcup_j P^{w_1}_{J_2}a_j$. Let $\overline{S} = (\overline{s}_{k,j})$ be the sadwich matrix of $(P^{w_1}_{J_2}, L_{J_1}, P^{w_2w_1}_{J_3})$ where $w_2 \in W, w_2(\Delta_{J_2}) = \Delta_{J_3}$, with respect to $G = \bigsqcup_j a_j^{-1}P^{w_1}_{J_2} = \bigsqcup_k P^{w_2w_1}_{J_3}c_k$. Consider also the sandwich matrix $\hat{S} = (\hat{s}_{k,i})$ of $(P_{J_1}, L_{J_1}, P^{w_2w_1}_{J_3})$ with respect to $G = \bigsqcup_i b_iP_{J_1} = \bigsqcup_k P^{w_2w_1}_{J_3}c_k$. With this notation we have

Theorem 4.1. *Assume that* $l(w_2w_1) = l(w_2) + l(w_1)$. *Then* $\overline{S}S = \hat{S}$.

The proof will be preceded by a sequence of auxiliary lemmas.

Lemma 4.2. *Assume that* $\alpha > 0, w_2(\alpha) < 0$ *and* $l(w_2w_1) = l(w_2)+l(w_1)$ *for some* $\alpha \in \Phi, w_1, w_2 \in W$. *Then* $w_1^{-1}(\alpha) > 0$.

Proof. We proceed by induction on $l(w_2)$. If $l(w_2) = 0$, then the claim is obvious. So suppose $l(w_2) > 0$. Then there exist $s_i, i \in I$ and $w_2' \in W$ such that $w_2 = s_iw_2'$ and $l(w_2) = l(w_2') + 1$. Assume first, that $w_2'(\alpha) > 0$. By the hypothesis $s_i(w_2'(\alpha)) < 0$. Hence $w_2'(\alpha) = \alpha_i$. This gives $\alpha = w_2'^{-1}(\alpha_i)$. So we have to prove that $w_1^{-1}w_2'^{-1}(\alpha_i) > 0$. By [1, Prop.2.2.8] this follows from the equality $l(w_1^{-1}w_2'^{-1}s_i) = l(w_1^{-1}w_2'^{-1}) + 1$. It remains to consider the case where $w_2'(\alpha) < 0$. Since $\alpha > 0, w_2'(\alpha) < 0$ and $l(w_2'w_1) = l(w_2') + l(w_1)$, we can apply the induction hypothesis to w_2' and w_1 to obtain $w_1^{-1}(\alpha) > 0$.

Lemma 4.3. *Assume, that* $w_1(\Delta_{J_1}) = \Delta_{J_2}$, $w_2(\Delta_{J_2}) = \Delta_{J_3}$ *and* $l(w_2w_1) = l(w_2) + l(w_1)$ *for some* $w_1, w_2 \in W$. *Then*
(a) $w_2U_{J_2}w_1 \subseteq U_{J_3}w_2w_1U_{J_1}$,
(b) $w_2P_{J_2}w_1 \subseteq P_{J_3}w_2w_1P_{J_1}$.

Proof. (a) By [1, Prop.2.5.11] we have $U = U_{w_0w_2}U_{w_2}$. Since U_{J_2} is a product of certain root subgroups we have also $U_{J_2} = (U_{J_2} \cap U_{w_0w_2})(U_{J_2} \cap U_{w_2})$. Hence

$$\begin{aligned} w_2U_{J_2}w_1 &= w_2(U_{J_2} \cap U_{w_0w_2})(U_{J_2} \cap U_{w_2})w_1 \\ &= (U_{J_2} \cap U_{w_0w_2})^{w_2^{-1}}(w_2w_1)(U_{J_2} \cap U_{w_2})^{w_1}. \end{aligned} \tag{3}$$

First we will prove that $(U_{J_2} \cap U_{w_0w_2})^{w_2^{-1}} \subseteq U_{J_3}$. $U_{J_2} \cap U_{w_0w_2}$ is a product of X_α where $\alpha > 0, \alpha \notin \Phi_{J_2}$ and $w_0w_2(\alpha) < 0$. This implies that $w_2(\alpha) > 0$ and $w_2(\alpha) \notin w_2(\Phi_{J_2}) = \Phi_{J_3}$. Hence $X_\alpha^{w_2^{-1}} = X_{w_2(\alpha)} \subseteq U_{J_3}$. This proves the desired inclusion. Next, we claim that $(U_{J_2} \cap U_{w_2})^{w_1} \subseteq U_{J_1}$. $U_{J_1} \cap U_{w_2}$ is a product of X_α with $\alpha > 0, \alpha \notin \Phi_{J_2}$ and $w_2(\alpha) < 0$. By Lemma 4.2 the first and the last condition imply that $w_1^{-1}(\alpha) > 0$. We have also $w_1^{-1}(\alpha) \notin w_1^{-1}(\Phi_{J_2}) = \Phi_{J_1}$. Hence $X_\alpha^{w_1} \subseteq U_{J_1}$, proving the claim. In view of (3), the two established inclusions imply that (a) holds.

(b) Now, using the first assertion we have

$$\begin{aligned} w_2P_{J_2}w_1 = w_2U_{J_2}L_{J_2}w_1 = w_2U_{J_2}w_1L_{J_2}^{w_1} &\subseteq \text{by (a)} \\ \subseteq U_{J_3}w_2w_1U_{J_1}L_{J_1} &\subseteq P_{J_3}w_2w_1P_{J_1}. \end{aligned}$$

Lemma 4.4. *Assume that* $w_1(\Delta_{J_1}) = \Delta_{J_2}, w_2(\Delta_{J_2}) = \Delta_{J_3}$ *and* $l(w_2w_1) = l(w_2)+ l(w_1)$ *for some* $w_1, w_2 \in W$. *Moreover, if* $w_2u_2w_1 = u_3w_2w_1u_1l$ *for some* $u_i \in U_{J_i}, l \in L_{J_1}$, *then* $l = 1$.

Proof. By Lemma 4.3(a) there exist $u_3' \in U_{J_3}$ and $u_1' \in U_{J_1}$ such that $w_2u_2w_1 = u_3'w_2w_1u_1'$. Hence by the hypothesis $u_3'w_2w_1u_1' = u_3w_2w_1u_1l$. This gives $u_1lu_1'^{-1} = (w_2w_1)^{-1}u_3^{-1}u_3'(w_2w_1)$. So we have $u_1lu_1'^{-1} \in P_{J_1} \cap U_{J_3}^{w_2w_1} = U_{J_1} \cap U_{J_3}^{w_2w_1}$, the equality following from [1, Prop.2.8.6 and Cor.2.8.8] since $(w_2w_1)^{-1}(\Delta_{J_3}) = \Delta_{J_1}$. Thus $l \in U_{J_1}$. Then $U_{J_1} \cap L_{J_1} = \{1\}$ implies that $l = 1$.

Lemma 4.5. *Assume that* $w_1(\Delta_{J_1}) = \Delta_{J_2}, w_2(\Delta_{J_2}) = \Delta_{J_3}$ *and* $l(w_2w_1) = l(w_2)+ l(w_1)$ *for some* $w_1, w_2 \in W$. *Then* $U_{J_3} \cap U_{J_1}^{(w_2w_1)^{-1}} \subseteq U_{J_2}^{w_2^{-1}}$.

Proof. The assertion is equivalent to

$$U \cap U^{(w_0)_{J_3}} \cap (U \cap U^{(w_0)_{J_1}})^{(w_2w_1)^{-1}} \subseteq (U \cap U^{(w_0)_{J_2}})^{w_2^{-1}}$$

Hence it is sufficient to show that

(i) $U \cap U^{(w_2w_1)^{-1}} \subseteq U^{w_2^{-1}}$,

(ii) $U^{(w_0)_{J_3}} \cap U^{(w_0)_{J_1}(w_2w_1)^{-1}} \subseteq U^{(w_0)_{J_2}w_2^{-1}}$.

To prove (i) first note that $U \cap U^{(w_2w_1)^{-1}} = U_{w_0(w_2w_1)^{-1}}$ is a product of X_α where $\alpha > 0, w_0(w_2w_1)^{-1}(\alpha) < 0$. Hence $(w_2w_1)^{-1}(\alpha) > 0$. If $w_2^{-1}(\alpha) < 0$ then by Lemma 4.2 (where α stands for $w_2^{-1}(-\alpha)$) $w_1^{-1}w_2^{-1}(\alpha) < 0$. This contradiction shows that $w_2^{-1}(\alpha) > 0$. Therefore $X_\alpha \subseteq U^{w_2^{-1}}$, which proves the first inclusion.

Next we transform the second inclusion to the form

$$U^{(w_0)_{J_3}} \cap U^{(w_2w_1)^{-1}(w_2w_1)(w_0)_{J_1}(w_2w_1)^{-1}} \subseteq U^{w_2^{-1}w_2(w_0)_{J_2}w_2^{-1}}.$$

This is equvalent to

$$U^{(w_0)_{J_3}} \cap U^{(w_2w_1)^{-1}(w_0)_{J_3}} \subseteq U^{w_2^{-1}(w_0)_{J_3}}.$$

Conjugating by $(w_0)_{J_3}$ we get (i). So the lemma follows.

Lemma 4.6. *Assume that* $w_1(\Delta_{J_1}) = \Delta_{J_2}, w_2(\Delta_{J_2}) = \Delta_{J_3}$ *and* $l(w_2w_1) = l(w_2) + l(w_1)$ *for some* $w_1, w_2 \in W$. *If moreover* $w_2u_2w_1 = u_3w_2w_1u_1$ *for some* $u_i \in U_{J_i}$ *then* $u_3 \in U_{J_2}^{w_2^{-1}}$.

Proof. By the proof of Lemma 4.3(a) there exist $u_1' \in U_{J_1}, u_3' \in U_{J_3} \cap U_{J_2}^{w_2^{-1}}$ such that $w_2u_1w_1 = u_3'w_2w_1u_1'$. Hence by the hypothesis $u_3(w_2w_1)u_1 = u_3'(w_2w_1)u_1'$. This gives $u_3'^{-1}u_3 = (w_2w_1)u_1'u_1^{-1}(w_2w_1)^{-1}$. Thus by Lemma 4.5 $u_3'^{-1}u_3 \in U_{J_3} \cap U_{J_1}^{(w_2w_1)^{-1}} \subseteq U_{J_2}^{w_2^{-1}}$. Since $u_3' \in U_{J_2}^{w_2^{-1}}$ we have $u_3 \in U_{J_2}^{w_2^{-1}}$.

Proof of Theorem 4.1. We have to show that $\sum_j \overline{s}_{k,j}s_{j,i} = \hat{s}_{k,i}$ in $K[L_{J_1}]$ for any k, i. First, assume that there exists j_0 with $\overline{s}_{k,j_0} \neq 0$ and $s_{j_0.i} \neq 0$. In this case $c_ka_{j_0}^{-1} \in P_{J_3}^{w_2w_1}P_{J_2}^{w_1}$ and $a_{j_0}b_i \in P_{J_2}^{w_1}P_{J_1}$. Hence $c_ka_{j_0}^{-1} = u_3^{w_2w_1}\overline{u}_2^{w_1}\hat{s}_{k,j_0}$ and $a_{j_0}b_i = s_{j_0,i}u_2^{w_1}u_1$ for some $u_i \in U_{J_i}, \overline{u}_2 \in U_{J_2}$. It follows that

$$c_kb_i = c_ka_{j_0}^{-1}a_{j_0}b_i = w_1^{-1}w_2^{-1}u_3w_2\overline{u}_2w_1\overline{s}_{k,j_0}s_{j_0,i}w_1^{-1}u_2w_1u_1.$$

On the other hand

$$c_kb_i \in P_{J_3}^{w_2w_1}P_{J_2}^{w_1}P_{J_1} = w_1^{-1}w_2^{-1}P_{J_3}(w_2P_{J_2}w_1)P_{J_1} \subseteq$$
$$\subseteq w_1^{-1}w_2^{-1}P_{J_3}(P_{J_3}w_2w_1P_{J_1})P_{J_1} = P_{J_3}^{w_2w_1}P_{J_1}$$

by Lemma 4.3(b). Thus $c_kb_i = w_1^{-1}w_2^{-1}\hat{u}_3w_2w_1\hat{u}_1\hat{s}_{k,i}$ for some $\hat{u}_1 \in U_{J_1}, \hat{u}_3 \in U_{J_3}$, which are independent of j. Comparing the obtained expressions for c_kb_i we come to

$$u_3w_2\overline{u}_2(\overline{s}_{k,j_0}s_{j_0,i})^{w_1^{-1}}u_2w_1u_1 = \hat{u}_3w_2w_1\hat{u}_1\hat{s}_{k,i}.$$

Hence $u_3w_2(\overline{u}_2u_2^*)w_1(\overline{s}_{k,j_0}s_{j_0,i})u_1 = \hat{u}_3w_2w_1\hat{u}_1\hat{s}_{k,i}$ where $u_2^* = u_2^{(w_1\overline{s}_{k,j_0}s_{j_0,i}w_1^{-1})^{-1}} \in U_{J_2}$. Therefore

$$w_2(\overline{u}_2u_2^*)w_1 = (u_3^{-1}\hat{u}_3)(w_2w_1)(\hat{u}_1\hat{s}_{k,i}u_1^{-1}\hat{s}_{k,i}^{-1})[\hat{s}_{k,i}(\overline{s}_{k,j_0}s_{j_0,i})^{-1}].$$

Lemma 4.4 implies that $\hat{s}_{k,i}(\overline{s}_{k,j_0}s_{j_0,i})^{-1} = 1$. Thus $\overline{s}_{k,j_0}s_{j_0,i} = \hat{s}_{k,i}$. For any j such that $\overline{s}_{k,j} \neq 0$ and $s_{j,i} \neq 0$ we have

$$w_2(\overline{u}_2u_2^*)w_1 = (u_3^{-1}\hat{u}_3)(w_2w_1)(\hat{u}_1\overline{s}_{k,i}u_1^{-1}\overline{s}_{k,i}^{-1}).$$

Lemma 4.6 implies $u_3^{-1}\hat{u}_3 \in U_{J_2}^{w_2^{-1}}$. Hence $u_3^{-1}\hat{u}_3 = w_2\tilde{u}_2^{-1}w_2^{-1}$ for some $\tilde{u}_2 \in U_{J_2}$. This gives $u_3 = \hat{u}_3\tilde{u}_2^{w_2^{-1}}$. So we have

$$\begin{aligned} c_k a_j^{-1} &= w_1^{-1}w_2^{-1}u_3 w_2\overline{u}_2 w_1\overline{s}_{k,j} = w_1^{-1}w_2^{-1}\hat{u}_3 w_2\tilde{u}_2 w_2^{-1}w_2\overline{u}_2 w_1\overline{s}_{k,j} \\ &= \hat{u}_3^{w_2w_1}(\tilde{u}_2\overline{u}_2)^{w_1}\overline{s}_{k,j} \in \hat{u}_3^{w_2w_1}P_{J_2}^{w_1}. \end{aligned}$$

Hence $a_j \in P_{J_2}^{w_1}(\hat{u}_3^{-1})^{w_2w_1}c_k$. Since $\hat{u}_3$ is independent of j, the index j is determined uniquely. Hence $\sum_j \overline{s}_{k,j}s_{j,i} = \overline{s}_{k,j_0}s_{j_0,i} = \hat{s}_{k,i}$.

It remains to consider the case where, for every j, $\overline{s}_{k,j} = 0$ or $s_{j,i} = 0$. We have to prove that $\hat{s}_{k,i} = 0$. If $\hat{s}_{k,i} \neq 0$ then there exist $\hat{u}_1 \in U_{J_1}$ and $\hat{u}_3 \in U_{J_3}$ such that $c_k b_i = w_1^{-1}w_2^{-1}\hat{u}_3 w_2 w_1\hat{u}_1\hat{s}_{k,i}$. Choose j with $a_j \in P_{J_2}^{w_1}(\hat{u}_3^{-1})^{w_2w_1}c_k$. Then $c_k a_j^{-1} \in P_{J_3}^{w_2w_1}P_{J_2}^{w_1}$, hence $\overline{s}_{k,j} \neq 0$. Moreover

$$\begin{aligned} a_j b_i &= (a_j c_k^{-1})(c_k b_i) = (a_j c_k^{-1})(\hat{u}_3^{w_2w_1})\hat{u}_1\hat{s}_{k,i} \\ &\in P_{J_2}^{w_1}(\hat{u}_3^{-1})^{w_2w_1}\hat{u}_3^{w_2w_1}\hat{u}_1\hat{s}_{k,i} \subseteq P_{J_2}^{w_1}P_{J_1}. \end{aligned}$$

This implies that $s_{j,i} \neq 0$. This contradiction completes the proof.

Our last aim is to obtain a useful decomposition of the sandwich matrix. Let $J \subseteq I$ and $\alpha \in \Delta\backslash\Delta_J$. We then define $w_{\alpha,J} \in W$ by $w_{\alpha,J} = (w_0)_{J\cup\{\alpha\}}(w_0)_J$. Recall that $(w_0)_{J\cup\{\alpha\}}$ and $(w_0)_J$ are the elements of maximal length in the Coxeter groups with simple root systems $\Delta_J \cup \{\alpha\}$ and Δ_J respectively. Let $J_1, J' \subseteq I, w \in W$ and $w(\Delta_{J_1}) = \Delta_{J'}$. Then by [1, Prop.10.7.2] w can be expressed in the form $w = w_{\alpha_k,J_k}\dots w_{\alpha_1,J_1}$ where $l(w) = l(w_{\alpha_k,J_k}) + \dots + l(w_{\alpha_1,J_1}), \alpha_1, \dots, \alpha_k \in \Delta$ and $J_1, \dots, J_k \subseteq I$ are such that $w_{\alpha_i,J_i}(\Delta_{J_i}) = \Delta_{J_{i+1}}$ and $J_{k+1} = J'$. Hence by Theorem 4.1 the sandwich matrix of $(P_{J_1}, L_{J_1}, P_{J'}^w) = (P_{J_1}, L_{J_1}, P_{J'}^{w_{\alpha_k,J_k}\dots w_{\alpha_1,J_1}})$ may be expressed as a product of matrices which are conjugate by some elements of W to sandwich matrices of type $(P_{J_1}, L_{J_1}, P_{J_2}^{w_{\alpha,J_1}})$ where $w_{\alpha,J_1}(\Delta_{J_1}) = \Delta_{J_2}, J_1, J_2 \subseteq I$. In particular, the problem of finding the inverse of the sandwich matrix is reduced to this case. We will investigate the sandwich matrix of the latter type. We start by choosing convenient coset representatives for parabolic subgroups.

Lemma 4.7. *Consider coset decompositions* $L_{J_1\cup\{\alpha\}} = \bigsqcup_i b_i(P_{J_1} \cap L_{J_1\cup\{\alpha\}}) = \bigsqcup_j (P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}})a_j$ *and* $G = \bigsqcup_k c_k P_{J_1\cup\{\alpha\}}$. *Then* G *is decomposed as follows:* $G = \bigsqcup_{k,i} c_k b_i P_{J_1} = \bigsqcup_{j,l} P_{J_2}^{w_{\alpha,J_1}} a_j c_l^{-1}$.

Proof. First we show that the cosets $c_k b_i P_{J_1}$ are disjoint (similarly, one shows that $P_{J_2}^{w_{\alpha,J_1}} a_j c_l^{-1}$ are disjoint). Assume that $c_{k_1}b_{i_1}P_{J_1} = c_{k_2}b_{i_2}P_{J_1}$. Then $b_{i_2}^{-1}c_{k_2}^{-1}c_{k_1}b_{i_1} \in P_{J_1}$. Hence $c_{k_2}^{-1}c_{k_1} \in b_{i_2}P_{J_1}b_{i_1}^{-1} \subseteq P_{J_1\cup\{\alpha\}}$. By the hypothesis we have $k_1 = k_2$. So $b_{i_2}^{-1}b_{i_1} \in P_{J_1} \cap L_{J_1\cup\{\alpha\}}$. This gives $i_1 = i_2$. Next, we prove that $G = \bigcup_{k,i} c_k b_i P_{J_1}$ and $G = \bigcup_{j,l} P_{J_2}^{w_{\alpha,J_1}} a_j c_l^{-1}$. If $g \in G$ then $g \in c_k P_{J_1\cup\{\alpha\}}$ for some k. Hence $g = c_k ul$ for some $u \in U_{J_1\cup\{\alpha\}}, l \in L_{J_1\cup\{\alpha\}}$. Similarly $l \in b_i(P_{J_1} \cap L_{J_1\cup\{\alpha\}})$ for some i, so $l = b_i p$ where $p \in P_{J_1} \cap L_{J_1\cup\{\alpha\}}$. Then we have $gP_{J_1} = c_k ulP_{J_1} = c_k ub_i pP_{J_1} = c_k b_i(b_i^{-1}ub_i)P_{J_1} = c_k b_i P_{J_1}$ since $b_i^{-1}ub_i \in U_{J_1\cup\{\alpha\}}$. Now, consider the second coset decomposition. Let $g \in G$. Then $g \in P_{J_1\cup\{\alpha\}}c_k^{-1}$ for some k. Hence $g = ulc_k^{-1}$ with $u \in U_{J_1\cup\{\alpha\}}$ and $l \in L_{J_1\cup\{\alpha\}}$. Similarly $l \in (P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}})a_j$ for some j. Thus $l = pa_j$ where $p \in P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}}$. Then we have $P_{J_2}^{w_{\alpha,J_1}}g = P_{J_2}^{w_{\alpha,J_1}}ulc_k^{-1} = P_{J_2}^{w_{\alpha,J_1}}upa_jc_k^{-1} = P_{J_2}^{w_{\alpha,J_1}}a_jc_k^{-1}$ provided that $U_{J_1\cup\{\alpha\}} \subseteq U_{J_2}^{w_{\alpha,J_1}}$. So it remains to prove the last inclusion. $U_{J_1\cup\{\alpha\}}$ is a product of root subgroups X_β where $\beta > 0$ and

$\beta \notin \Phi_{J_1\cup\{\alpha\}}$. Hence $(w_0)_{J_1}(\beta) > 0$. This implies that $(w_0)_{J_1\cup\{\alpha\}}((w_0)_{J_1}(\beta)) > 0$ since $(w_0)_{J_1}(\beta) \notin \Phi_{J_1\cup\{\alpha\}}$. Therefore $w_{\alpha,J_1}(\beta) > 0$. Moreover $w_{\alpha,J_1}(\beta) \notin \Phi_{J_2}$ since $w_{\alpha,J_1}(\Delta_{J_1}) = \Delta_{J_2}$. So $X_\beta^{w_{\alpha,J_1}^{-1}} \subseteq U_{J_2}$. This gives $X_\beta \subseteq U_{J_2}^{w_{\alpha,J_1}}$. Thus the lemma follows.

Let $D = (d_{(j,l),(k,i)})$ be the sandwich matrix of $(P_{J_1}, L_{J_1}, P_{J_2}^{w_{\alpha,J_1}})$ with respect to the following coset decompositions $G = \bigsqcup_{k,i} c_k b_i P_{J_1} = \bigsqcup_{j,l} P_{J_2}^{w_{\alpha,J_1}} a_j c_l^{-1}$. Let $E = (e_{j,i})$ be the sandwich matrix of the monoid defined by $(P_{J_1} \cap L_{J_1\cup\{\alpha\}}, L_{J_1}, P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}})$ (with the group of units $G = L_{J_1\cup\{\alpha\}}$) with respect to the coset decompositions $L_{J_1\cup\{\alpha\}} = \bigsqcup_i b_i(P_{J_1} \cap L_{J_1\cup\{\alpha\}}) = \bigsqcup_j (P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}})a_j$. With this notation the structure of D can be described as follows.

Proposition 4.8. *We have*
(a) $d_{(j,l),(k,i)} = 0$ *for* $k \neq l$,
(b) $d_{(j,k),(k,i)} = e_{j,i}$.
That is, D has a block diagonal form with the diagonal blocks equal to E.

Proof. Assume that $d_{(j,l),(k,i)} \neq 0$. Then $a_j c_l^{-1} c_k b_i \in P_{J_2}^{w_{\alpha,J_1}} P_{J_1}$. Hence $c_l^{-1} c_k \in a_j^{-1} P_{J_2}^{w_{\alpha,J_1}} P_{J_1} b_i^{-1} \subseteq P_{J_1\cup\{\alpha\}}$ since $P_{J_2}^{w_{\alpha,J_1}} P_{J_1} \subseteq P_{J_1\cup\{\alpha\}}$. This implies that $k = l$, so (a) is proved . This gives $a_j b_i \in P_{J_2}^{w_{\alpha,J_1}} P_{J_1}$. Hence $a_j b_i = u_2^{w_{\alpha,J_1}} u_1 d_{(j,l),(k,i)}$ for some $u_i \in U_{J_i}$. Since $u_2^{w_{\alpha,J_1}}, u_1 \in P_{J_1\cup\{\alpha\}}$ then $u_2^{w_{\alpha,J_1}} = \overline{u}_2 l, u_1 = \overline{u}_1 \overline{l}$ for some $\overline{u}_i \in U_{J_1\cup\{\alpha\}}$ and $l, \overline{l} \in L_{J_1\cup\{\alpha\}}$. By the last part of the proof of Lemma 4.7 we have $U_{J_1\cup\{\alpha\}} \subseteq U_{J_2}^{w_{\alpha,J_1}}$. Hence $l \in U_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}}$. Similarly $\overline{l} \in U_{J_1} \cap L_{J_1\cup\{\alpha\}}$ since $U_{J_1\cup\{\alpha\}} \subseteq U_{J_1}$. So we have $a_j b_i = \overline{u}_2 l \overline{u}_1 \overline{l} d_{(j,l),(k,i)} = \overline{u}_2 (l \overline{u}_1 l^{-1})(l \overline{l} d_{(j,l),(k,i)})$. Since $a_j b_i \in L_{J_1\cup\{\alpha\}}$ then $\overline{u}_2 (l \overline{u}_1 l^{-1}) = 1$. Hence

$$a_j b_i = l \overline{l} d_{(j,l),(k,i)} \in (P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}})(P_{J_1} \cap L_{J_1\cup\{\alpha\}})$$

with $l \in U_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}}, \overline{l} \in U_{J_1} \cap L_{J_1\cup\{\alpha\}}$. This proves that $d_{(j,l),(k,i)} = e_{j,i}$ $(k = l)$ in the case where $d_{(j,k),(k,i)} \neq 0$. If $d_{(j,k),(k,i)} = 0$ then $e_{j,i} = 0$ by the definition, so we have also $d_{(j,k),(k,i)} = e_{j,i}$. This completes the proof of (b).

Since $P_{J_1} \cap L_{J_1\cup\{\alpha\}}, P_{J_2}^{w_{\alpha,J_1}} \cap L_{J_1\cup\{\alpha\}}$ are maximal parabolic subgroups of $L_{J_1\cup\{\alpha\}}$ with common Levi factor L_{J_1}, they are also opposite. Therefore, by the paragraph preceding Lemma 4.7 together with this lemma and Proposition 4.8, one is reduced to considering monoids $M = M(G, P, P^-, L)$ where P and P^- are opposite and maximal parabolic subgroups of G.

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